Existence of a codimension-2 point at the threshold of binary-fluid convection between rigid, impermeable boundaries

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Numerical calculation of the linear instability to oscillatory and stationary convection in binary-fluid mixtures between rigid, impermeable boundaries shows that the critical wave number and frequency of the oscillatory solution jump discontinuously as the threshold Rayleigh number is followed as a function of fluid parameters, contrary to a recent claim of B. J. A. Zielinska and H. R. Brand [Phys. Rev. A **35**, 4349 (1987)]. A degenerate codimension-2 point is found only if the wave numbers of the onset solutions are constrained.

Until recently, results for the parameters describing the linear instability to oscillatory convection in binary-fluid mixtures (e.g., the critical Rayleigh number and the wave number, frequency, and group velocity of the waves at onset) were not reliably known for the common experimental arrangement of rigid, impermeable upper and lower boundaries. Instead, only results from uncontrolled trial-function methods were available.[1] On the other hand, detailed experimental measurements are now available, and it seems useful to have reliable theoretical values to compare with experiment. Zielinska and Brand[2] recently presented results for the critical wave number, frequency, and Rayleigh number. We have also calculated these results, together with the group velocity, growth rate, dispersion, and other quantities for a wide range of fluid parameters. These will be compared in detail with experiments elsewhere. In this note we wish to correct a point that is of considerable conceptual interest, although perhaps hard to measure experimentally. The question is the nature of the "codimension-2" point, where the oscillatory instability disappears in favor of the stationary one. We find in our full numerical calculation, and in agreement with the conclusion of Linz and Lücke[3] arrived at by approximate mode truncation methods, that (in a sense to be carefully defined) there *are* wave number and frequency jumps in the codimension-2 region. Precisely, if we always follow the *lowest* Rayleigh number for instability as the separation ratio ψ varies for a given Lewis number L, allowing the wave number of the onset solution to change appropriately, the frequency of the onset solution jumps discontinuously to zero at some ψ_C, together with a jump in the critical wave number. In this sense the degenerate codimension-2 point is not reached if the wave number is allowed to adjust to give the lowest threshold. If the wave number k is held fixed, then indeed the frequency goes continuously to zero at $\psi^*(k)$, $R^*(k)$ depending on k.

Our discussion of the codimension-2 region is summarized in terms of the second-order equation[4] for the amplitude W_k of the onset mode for some wave number k,

$$\ddot{W}_k+\alpha(k)\dot{W}_k+\beta(k)=0\ , \tag{1}$$

where α,β also depend on the system parameters Rayleigh number R, separation ratio ψ, Lewis number L (the ratio of solute diffusivity to thermal conductivity) and Prandtl number σ (the ratio of thermal diffusivity to kinematic viscosity). The oscillatory instability is given by $\alpha=0$, $\beta>0$, and the stationary instability by $\beta=0$, $\alpha>0$. The codimension-2 point for a fixed wave number k is given by tuning two system parameters (conceptually the Rayleigh number to R^* and the separation ration to ψ^* at fixed Lewis number L) to make $\alpha(k)=\beta(k)=0$. If the wave number is allowed to vary, and follows the value giving the least onset Rayleigh number, a degenerate bifurcation point ($\alpha=\beta=0$) will only be reached if $\alpha(k)$ and $\beta(k)$ are maximized (for given R,ψ) by the same value of k. There is no reason for this to happen in general; this requires the tuning of a third parameter, and for this particular problem seems to correspond to tuning L to zero. To understand the general case it is convenient to expand $\alpha(k)$ and $\beta(k)$, and to display for each ψ the dependence on the Rayleigh number, which is the parameter that may be most easily changed in experiments. Equation (1) may be approximated[5]

$$\ddot{W}_k+\bar{\alpha}[r_O-\xi_O^2(k-k_O)^2]\dot{W}_k+\bar{\beta}[r_S-\xi_S^2(k-k_S)^2]W_k=0\ , \tag{2}$$

with $k_S(\psi),k_O(\psi)$ the wave numbers maximizing α and β and r_S giving $[R-R_{SC}(\psi)]$ etc. with $R_{SC}(\psi),R_{OC}(\psi)$ the minimum Rayleigh numbers for stationary and oscillatory convection. We have for convenience assumed k_O and k_S are close so that expanding the k dependence to second order about each minimum is adequate. Solving (2) for a time dependence $e^{\lambda t}$ may be used to understand the bifurcations presented below.

We have calculated the linear instability to stationary and oscillatory convecting states using the same numerical approach described by Zielinska and Brand.[2] We find in general $k_S\neq k_O$. However, k_S does approach k_O for Lewis number L going to zero, and careful numerical work is needed to investigate the small values of the Lewis number reported in Ref. 2. To show the effect more easily we display results for a much larger value of

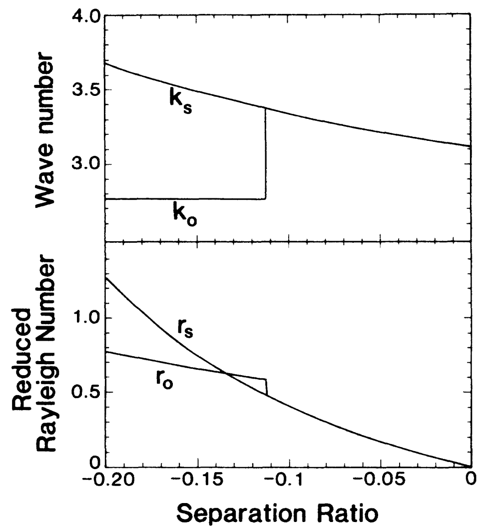

FIG. 1. Wave numbers k_O and k_S at the minimum threshold Rayleigh numbers for the oscillatory and stationary instabilities (upper graph) and values of the reduced Rayleigh number $r_O=(R_O-R_c)/R_c$ and $r_S=(R_S-R_c)/R_c$ (with R_c the critical Rayleigh number for pure fluid convection) at these minima (lower graph) as a function of the separation ratio for $L=0.8$, $\sigma=10$.

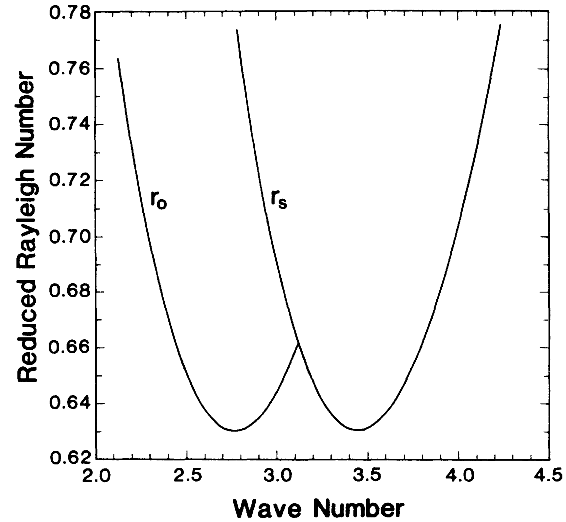

FIG. 2. Reduced Rayleigh numbers $r_S(k)=[R_S(k)-R_c]/R_c$ and $r_O(k)=[R_O(k)-R_c]/R_c$ for stationary and oscillatory instabilities at the separation ratio $\psi=-0.13(5)$ for which the threshold values coincide. Lewis number and Prandtl number are the same as in Fig. 1.

$L=0.8$. Figure 1 shows the critical wave numbers [i.e., at the quadratic minima of $R_S(k)$ and $R_O(k)$] and corresponding critical Rayleigh numbers. Note that the oscillatory instability [more precisely, the quadratic minimum in $R_O(k)$] disappears at $\psi=-0.11$ with wave number $k_O=2.76$ and Rayleigh number $1.58R_c$ (with R_c the critical Rayleigh number for pure-fluid convection). At this ψ the stationary threshold occurs at wave number $k_S=3.37$ and $R_S=1.47R_c$. The frequency of the oscillatory solution at k_O goes to zero continuously at this point. On the other hand, the critical Rayleigh numbers for oscillatory and stationary convection coincide for a more negative value of $\psi=-0.13(5)$ but with different wave numbers $k_S=3.44$, $k_O=2.76$. Curves for the onset Rayleigh numbers for this value of ψ are shown in Fig. 2. If we always choose the *lowest* onset Rayleigh number the frequency of the oscillatory solution is finite at the switch over, and the wave number jumps.

The jumps for small L are much smaller. For $L=0.04$ and $\sigma=0.75$ we find the wave number jumps when the critical Rayleigh numbers coincide (as in Fig. 2) to be from $k_O=3.07$ to $k_S=3.15$. For $L=0.02$ and $\sigma=17$ the jump is from $k_O=3.10(5)$ to $k_S=3.13$. We can show analytically that the wave-number jump is proportional to L for small Lewis number, and have also calculated the proportionality constant by a numerical scheme that is independent of L, and so it does not suffer from precision problems.[5] Clearly these are rather small changes in the wave number. However, conceptually it is important that in experiments in which the wave number is not controlled, the degenerate codimension-2 point may not be observed. Also when we consider deriving a spatially dependent degenerate amplitude equation based on Eq. (2) there will be extra linear spatial derivative terms from the fact that $k_S \neq k_O$; the equation suggested by Brand *et al.*[4] does not apply in this limit. Such extra terms might be expected to become important in system sizes greater than about $\frac{1}{2}k_S/|k_O-k_S|$ roll diameters.

Editor's note. The reader should note that an Erratum to Ref. 2 by B. J. A. Zielinska and H. R. Brand, which was submitted subsequent to the submission of this paper, has recently appeared [Phys. Rev. A **37**, 1786 (1988)].

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